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A CENTRAL LIMIT THEOREM FOR MARTINGALES AND AN APPLICATION TO BRANCHING PROCESSES

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A functional central limit theorem is obtained for martingales which are not uniformly asymptotically negligible but grow at a geometric rate. The function space is not the usual $C[0, 1]$ or $D[0, 1]$ but $\mathbf{R}^{\mathbf{N}}$, the space of all real sequences and the metric used leads to a non-separable metric space.

The main theorem is applied to a martingale obtained from a supercritical Galton–Watson branching process and as simple corollaries the already known central limit theorems for the Harris and Lotka–Nagaev estimators of the mean of the offspring distribution, are obtained.

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Functional central limit theorems	invariance principles
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	estimators of the mean of the offspring distribution

1. Introduction

In this paper we obtain a functional central limit theorem for martingales (MG's), and apply it to obtain a functional central limit theorem for a super-critical Galton–Watson branching process. The function space involved is $\mathbf{R}^{\mathbf{N}}$, the space of all real sequences. The reason for this setting rather than the usual $C[0, 1]$ or $D[0, 1]$ is that we are concerned with MG differences which are not uniformly asymptotically negligible but rather increase geometrically. In $\mathbf{R}^{\mathbf{N}}$ no metric which induces the product topology can be of interest since the finite-dimensional distributions are then convergence-determining for the Borel σ-field and a functional limit theorem in this setting could produce nothing more than can be given by convergence of the finite-dimensional distributions. The metric used here produces a nonseparable metric space.

The MG limit theorem is an extension of a one-dimensional limit theorem of Feigin [5] which has also beeen noted in Heyde and Feigin [8]. The setting and metric are taken from Heyde and Brown [7] who prove a functional limit theorem for a supercritical Galton–Watson branching process $Z_0 = 1, Z_1, Z_2, \ldots$ also. Their theorem concerns random sequences based on $(Z_n, Z_{n+1}, Z_{n+2}, \ldots)$. Here the sequences are functions of $(Z_0, \ldots, Z_n)$ which makes the theorem of use in inference. In particular we can readily obtain central limit theorems for the Lotka–Nagaev estimator of the mean and the Harris (sometimes inaccurately termed maximum likelihood) estimator of the mean.

Finally it should be noted that all the results are of the stable or mixing type (see for instance Rényi [10] and [11], or Richter [12] and [13]) and thus allow treatment of joint distributions and random indexing (Billingsley [2] Theorem 4.5 and section 17, Katai and Mogyoródi [9], Fischler [6]) although care is required to avoid problems caused by non-separability of the metric space.

2. The space $\mathbf{R}^{\mathbf{N}}$

Let $\mathbf{R}^{\mathbf{N}}$ denote the space of real sequences $x = (x_0, x_1, x_2, \ldots)$. For α and β real let

$$\rho(\alpha, \beta) = |\alpha - \beta|/(1 + |\alpha - \beta|)$$

and then define the metrics d_1, d_2 and d_3 on $\mathbf{R}^{\mathbf{N}}$, by

$$d_1(x, y) = \sup_{n \geq 0} \rho\left(\sum_{j=0}^{n} x_j, \sum_{j=0}^{n} y_j\right),$$

$$d_2(x, y) = \sup \rho(x_n, y_n)$$

and

$$d_3(x, y) = \sum_{n=0}^{\infty} 2^{-n} \rho(x_n, y_n).$$

The Borel σ-fields generated by d_1, d_2 and d_3 in $\mathbf{R}^{\mathbf{N}}$ will be denoted respectively $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{B}$. It can easily be shown that $(\mathbf{R}^{\mathbf{N}}, d_1)$ and $(\mathbf{R}^{\mathbf{N}}, d_2)$ are complete but not separable metric spaces while $(\mathbf{R}^{\mathbf{N}}, \mathcal{B})$ is a complete separable metric space. We have also $d_1(x, y) \leq 2d_2(x, y) \leq 4d_3(x, y)$, $\forall x, y \in \mathbf{R}^{\mathbf{N}}$ so that $\mathcal{D}_1 \supset \mathcal{D}_2 \supset \mathcal{B}$. The metric d_3 induces the product topology on $\mathbf{R}^{\mathbf{N}}$ and $\mathcal{B}$ is the usual σ-field generated by the cylinder sets. The Kolmogorov extension theorem allows us to assert the existence of measures on $(\mathbf{R}^{\mathbf{N}}, \mathcal{B})$ having a specified consistent set of finite-dimensional distributions. Suppose we have such a measure Q. It is necessary for this paper to see how it might be extended to $(\mathbf{R}^{\mathbf{N}}, \mathcal{D}_2)$ or even $(\mathbf{R}^{\mathbf{N}}, \mathcal{D}_1)$. If we have a sequence of sets with d_1-compact closure $D_1 \subset D_2, \ldots, D_i \in \mathcal{B}$ such that $\mathcal{D}_1 \cap D_i = \mathcal{B} \cap D_i$ for all i and $Q(D_i) \to 1$ as $i \to \infty$ then we may define $Q(D)$ for any $D \in \mathcal{D}_1$ as

$$Q(D) = \lim_{i \to \infty} Q(D_i \cap D)$$

and by Prohorov's Theorem (Theorem 6.1 of Billingsley [2]) Q is then a measure on $(\mathbf{R}^{\mathbf{N}}, \mathcal{D}_1)$ and hence on $(\mathbf{R}^{\mathbf{N}}, \mathcal{D}_2)$ also. It is clear that such an extension is unique on $(\mathbf{R}^{\mathbf{N}}, \mathcal{D}_1)$. The sets D_i are chosen to be of the form $B_M \cap C_r$ for particular $M \in \mathbf{R}^+$ and positive integral r where

$$B_m = \left\{x : \sup_{s \geq 0} \left| \sum_{j=0}^{s} x_j \right| \leq M \right\} \tag{2.1}$$

and

$$C_r = \left\{x : \sup_{s \geq k} \left| \sum_{j=k}^{s} x_j \right| \leq p^{-k/3} \text{ for all } k \geq r \right\} \tag{2.2}$$

for some $p > 1$ to be chosen later. Any set of the form $B_m \cap C_r$ has compact closure since any sequence belonging to it has a limit. The requirement that $\mathcal{D}_1 \cap D_i = \mathcal{B} \cap D_i$ follows from the fact that d_1 and d_3 generate the same topology on $B_m \cap C_r$. Suppose $A \subset B_m \cap C_r$ is d_1-open and let $x \in A$. Then there exists $\eta > 0$ such that $d_1(x, y) < \eta$ implies $y \in A$ and without loss of generality $\eta < \frac{1}{2}$. Now choose k such that $p^{-k/3} < \eta/2$ and consider any y such that $d_3(x, y) < 2^{-k-1}\eta$. We have

$$\begin{aligned} d_1(x, y) &= \sup_{n \geq 0} \rho\left(\sum_{k=0}^{n} x_j, \sum_{j=0}^{n} y_j\right) \\ &\leq \sum_{j=0}^{k} \rho(x_j, y_j) + \sup_{s \geq k} \rho\left(\sum_{k=k}^{s} x_j, \sum_{j=k}^{s} y_j\right) \\ &\qquad \text{(see Chung [3] p. 68)} \\ &\leq 2^k \sum_{j=0}^{k} \rho(x_j, y_j) 2^{-j} + p^{-k/3} \\ &\leq 2^k d_3(x, y) + p^{-k/3} \\ &< \eta, \end{aligned}$$

so that $y \in A$ and A is d_3-open as required.

3. A martingale central limit theorem

Let $\{S_n, \mathcal{F}_n ; n \leq 0\}$ be a martingale defined on a probability space $(\Omega, \mathcal{A}, P)$ with $S_n = \sum_{k=0}^{n} X_k$, $s_n^2 = \mathbf{E} S_n^2 < \infty$ and $V_n^2 = \sum_{k=0}^{n} \mathbf{E}\{X_k^2 \mid \mathcal{F}_{k-1}\}$ where $\mathcal{F}_{-1} = \{\phi, \Omega\}$.

For $p > 1$ and $a > 0$ define random sequences in $\mathbf{R}^{\mathbf{N}}$, ξ and ξ_n for $n \geq 1$, by

$$\xi_n = (\xi_{n0}, \xi_{n1}, \ldots), \tag{3.1}$$

$$\xi_{nj} = \begin{cases} X_{n-j}/(as_n), & j = 0, 1, 2, \ldots, n, \\ 0, & j > n. \end{cases} \tag{3.2}$$

and

$$\xi = (\eta Y_0, \eta Y_1, \ldots) \tag{3.3}$$

where $\eta \geq 0$ and is distributed independently of $Y_0, Y_1, \ldots$. The Y_j are also mutually independent with Y_j distributed as $\mathrm{N}(0, p^{-j})$. Further, set

$$\zeta_n = (\zeta_{n0}, \zeta_{n1}, \ldots) = s_n V_n^{-1} \xi_n \tag{3.4}$$

and

$$\zeta = (Y_0, Y_1, \dots). \tag{3.5}$$

Theorem 1. *Suppose that for some $p > 1$, $a > 0$ and η a random variable $\eta \geqslant 0$ a.s. we have*

$$s_n^{-2} V_n^2 \xrightarrow{P} \eta^2, \quad \textit{with } \mathbf{E}\eta^2 < \infty, \tag{3.7}$$

$$s_n^{-2} s_{n-r}^2 \to p^{-r} \quad \textit{for } r = 0, 1, 2, \dots \tag{3.8}$$

and

$$\mathbf{E}\{e^{itX_n/(aV_n)} \mid \mathcal{F}_{n-1}\} \xrightarrow{P} \exp(-t^2/2) \textit{ on } \{\eta > 0\}, \tag{3.9}$$

then for every $D \in \mathcal{D}_1$ such that $\mathbf{P}(\xi \in \partial D) = 0$ and every $A \in \mathcal{A}$,

$$\mathbf{P}(\{\xi_n \in D\} \cap A \mid \eta > 0) \to \mathbf{P}(\xi \in D)\mathbf{P}(A \mid \eta > 0) \tag{3.10}$$

and for every $D \in \mathcal{D}_1$ such that $\mathbf{P}(\zeta \in \partial D) = 0$ and every $A \in \mathcal{A}$

$$\mathbf{P}(\{\zeta_n \in D\} \cap A \mid \eta > 0) \to \mathbf{P}(\zeta \in D)\mathbf{P}(A \mid \eta > 0). \tag{3.11}$$

The following Lemma shows that (3.9) implies an apparently more stringent condition.

Lemma 1. *Conditions* (3.7), (3.8) *and* (3.9) *imply for each fixed integer $j \leqslant r$ for $r = 0, 1, 2, \dots$.*

$$\mathbf{E}\{e^{itX_{n-j}/Q_{n,r}} \mid \mathcal{F}_{n-j-1}\} \xrightarrow{P} \exp(-t^2 p^{-j}/2) \textit{ on } \{\eta > 0\}, \tag{3.12}$$

where $Q_{n,r} = a s_n V_{n-r-1} s_{n-r-1}^{-1}$.

Proof. By (3.8) and (3.7)

$$Q_{n,r} a^{-1} V_{n-j}^{-1} = s_n V_{n-r-1} s_{n-r-1}^{-1} V_{n-j}^{-1} \xrightarrow{P} p^{-j/2} \text{ on } \{\eta > 0\}$$

so that

$$e^{itX_{n-j}/Q_{n,r}} - e^{itp^{-j/2}X_{n-j}/(aV_{n-j})} \xrightarrow{P} 0 \quad \text{on } \{\eta > 0\}$$

and by boundedness

$$\mathbf{E}\{e^{itX_{n-j}/Q_{n,r}} \mid \mathcal{F}_{n-j-1}\} - \mathbf{E}\{e^{itp^{-j/2}X_{n-j}/(aV_{n-j})} \mid \mathcal{F}_{n-j-1}\} \xrightarrow{P} 0 \quad \text{on } \{\eta > 0\}.$$

Then (3.12) follows from (3.9).

Remark. A more stringent condition than (3.7), under which the results of Theorem 1 still obtain, is

$$s_n^{-2}V_n^2 \xrightarrow{L_1} \eta^2. \tag{3.12}$$

This is only slightly more stringent, since if (3.7) holds and $E\eta^2 = 1$, we have from Chung [3], Theorem 4.5.4 that (3.12) is satisfied.

4. Convergence of finite-dimensional distributions

Put

$$\phi_n = \phi(t_0, \dots, t_r; A) = \int_{A \cap \{V_{n-r-1}s_{n-r-1}^{-1} > \varepsilon\}} \exp\left(\mathrm{i}\sum_{j=0}^{r} t_j X_{n-j}/Q_{n,r}\right) \mathrm{d}P.$$

Suppose we can show that

$$\phi_n \to \mathbf{P}(A \cap \{\eta > \varepsilon\})\exp(-\tfrac{1}{2}[t_0^2 + p^{-1}t_1^2 + \cdots + p^{-r}t_r^2]) \tag{4.1}$$

for each $m, r, t_0, \dots, t_r$ and $A \in \mathscr{F}_m$ and each ε a continuity point of η. Then by the continuity theorem for characteristic functions and the fact that

$$Q_{n,r}(aV_n)^{-1} = V_{n-r-1}s_n s_{n-r-1}^{-1} V_n^{-1} \xrightarrow{P} 1 \quad \text{on } \{\eta > 0\}$$

we have for any m and $A \in \mathscr{F}_m$ and any r and $(r+1)$-dimensional Borel set B

$$\mathbf{P}(\{(\zeta_{n0}, \dots, \zeta_{nr}) \in B\} \cap A \cap \{\eta > \varepsilon\}) \to \mathbf{P}((Y_0, \dots, Y_r) \in B)P(A \cap \{\eta > \varepsilon\})$$

and hence

$$\mathbf{P}(\{(\zeta_{n0}, \dots, \zeta_{nr}) \in B\} \cap A \mid \eta > 0) \to \mathbf{P}((Y_0, \dots, Y_r) \in B)\mathbf{P}(A \mid \eta > 0).$$

Then from Theorem 2 of Rényi [10],

$$\mathbf{P}(\{(\zeta_{n0}, \dots, \zeta_{nr}) \in B\} \cap A \mid \eta > 0) \to \mathbf{P}((Y_0, \dots, Y_r) \in B)\mathbf{P}(A \mid \eta > 0) \tag{4.2}$$

for any $A \in \mathscr{A}$ and any $(r+1)$-dimensional Borel set B, which is the required convergence of finite-dimensional distributions for the sequences $\{\zeta_n\}$.

From an $(r+1)$-dimensional extension of Theorem 2 of Katai and Mogyoródi [9] (alternatively see Fischler [6])

$$\mathbf{P}(\{(\xi_{n0}, \dots, \xi_{nr}) \in B\} \cap A \mid \eta > 0) \to \mathbf{P}((\eta Y_0, \dots, \eta Y_r) \in B)\mathbf{P}(A \mid \eta > 0) \tag{4.3}$$

for every $A \in \mathscr{A}$ and $(r+1)$-deminsional Borel set B for which $\mathbf{P}((\eta Y_0, \dots, \eta Y_r) \in \partial B) = 0$. The convergence in (4.3) is the required convergence of the finite-dimensional distributions of $\{\xi_n\}$. We need only prove (4.1).

Set

$$A_j = \exp\{\mathrm{i}t_j X_{n-j}/Q_{n,r}\}, \quad j = 0, 1, \dots, r,$$

and

$$B_j = \left(\prod_{k=0}^{j-1} A_k\right) I(V_{n-r-1}s_{n-r-1}^{-1} > \varepsilon)\exp\left(-\tfrac{1}{2}\sum_{k=j+1}^{r} t_k^2 p^{-k}\right) [A_j - \exp(-\tfrac{1}{2}t_j^2 p^{-j})].$$

Then for $j = 0, 1, \ldots, r$ and sufficiently large n,

$$\begin{aligned}\mathbf{E}\{B_j \mid \mathscr{F}_{n-j-1}\} &= \left(\prod_{k=0}^{j-1} A_k\right) I(V_{n-r-1}s_{n-r-1}^{-1} > \varepsilon)\exp\left(-\tfrac{1}{2}\sum_{k=j+1}^{r} t_k^2 p^{-k}\right) \\ &\quad \times [\mathbf{E}\{A_j \mid \mathscr{F}_{n-j-1}\} - \exp(-\tfrac{1}{2}t_j^2 p^{-j})] \qquad (4.4) \\ &\xrightarrow{P} 0 \qquad \text{(by Lemma 1).}\end{aligned}$$

Now

$$|\phi_n - P(A \cap \{V_{n-r-1}s_{n-r-1}^{-1} > \varepsilon\})\exp(-\tfrac{1}{2}[t^2 + p^{-1}t_1^2 + \cdots + p^{-r}t_r^2])| =$$

$$= \left| \int_{A \cap \{V_{n-r-1}s_{n-r-1}^{-1} > \varepsilon\}} \sum_{j=0}^{r} B_j \, \mathrm{d}P \right| \leq \mathbf{E}\left(\left| \sum_{j=0}^{r} \{B_j \mid \mathscr{F}_{n-j-1}\} \right| \right) \to 0.$$

by (4.4), since $|B_j| \leq 2$.

5. Completion of proof of Theorem 1

The proof of Theorem 1 would commonly be completed by proving tightness, but it seems difficult to establish this directly for the sequences $\{\xi_n\}$ and $\{\zeta_n\}$. Instead we consider a double array $\{\xi_n^{(u)}\}$ and apply Theorem 4.2 of Billingsley [2]. Although this theorem is given under the restriction that the random elements lie in a separable metric space this is only necessary to ensure measurability of the function $d_1(\xi_n, \xi_n^{(u)})$ and since ξ_n and $\xi_n^{(u)}$ as defined later have a restricted range this cause no problem. Define $\xi_n^{(u)}$ for each $n = 1, 2, \ldots$ and each $n = 0, 1, 2, \ldots$ by

$$\xi_n^{(u)} = (\xi_{n0}^{(u)}, \xi_{n1}^{(u)}, \ldots),$$

where

$$\xi_{nj}^{(u)} = \begin{cases} X_{n-j}/as_n, & j = 0, 1, \ldots, u \wedge n, \\ 0, & j > u \wedge n. \end{cases}$$

It is obvious that for each fixed u the finite dimensional distributions of $\{\xi_n^{(u)}\}$ converge in the manner of (4.3) to those of $\xi^{(n)}$ where

$$\xi^{(n)} = (\eta Y_0, \eta Y_1, \ldots, \eta Y_u, 0, \ldots).$$

In addition, by convergence of the finite-dimensional distributions there exists an M such that for $\varepsilon > 0$ and

$$B_M = \left\{x \in \mathbf{R}^{\mathbf{N}} : \sup_{s \geq 0} \left| \sum_{j=0}^{s} x_j \right| \leq M \right\} \tag{5.1}$$

and

$$C_{u+1} = \left\{X \in \mathbf{R}^{\mathbf{N}}: \sup_{s \geq k} \left| \sum_{j=k}^{s} x_j \right| \leq p^{-k/3} \text{ for all } k \geq u+1 \right\} \tag{5.2}$$

we have

$$\mathbf{P}(\xi_n^{(u)} \in B_M \cap C_{u+1}) > 1 - \varepsilon$$

so the sequence $\{\xi_n^{(u)}\}$ is tight and thus for any $D \in \mathscr{D}_1$ for which $\mathbf{P}(\xi^{(n)} \in \partial D) = 0$ and every $A \in \mathscr{A}$,

$$\mathbf{P}(\{\xi^{(u)} \in D\} \cap A \mid \eta > 0) \to \mathbf{P}(\xi^{(u)} \in \partial D)\mathbf{P}(A \mid \eta > 0). \tag{5.3}$$

Now the finite-dimensional distributions of $\{\xi^{(u)}\}$ converge to those of ξ (again in the manner of (4.3)). We show $\{\xi^{(u)}\}$ is tight so that for any $D \in \mathscr{D}_1$ with $\mathbf{P}(\xi \in \partial D) = 0$ and any $A \in \mathscr{A}$.

$$\mathbf{P}(\{\xi^{(u)} \in D\} \cap A \mid \eta > 0) \to \mathbf{P}(\xi \in D)\mathbf{P}(A \mid \eta > 0). \tag{5.4}$$

Conditioning on η we have by Kolmogorov's inequality, for any $C > 0$

$$\begin{aligned}
\mathbf{P}\left(\sup_{s \geq t \geq k} \left| \sum_{j=k}^{t \wedge n} \eta Y_j \right| > C \mid \eta \right) &\leq C^{-2}\eta^2 \sum_{j=k}^{s \wedge u} \mathbf{E} Y_j^2 \\
&\leq C^{-2}\eta^2 \sum_{j=k}^{u} p^{-j} \quad \text{for all } s, \\
&= C^{-2}\eta^2 p^{-k}(1-p^{-1})(1-p^{-u-1})^{-1} \\
&\leq C^{-2}\eta^2 p^{-k}
\end{aligned}$$

so that

$$\mathbf{P}\left(\sup \left| \sum_{j=k}^{s \wedge u} \eta Y_j \right| > C \right) \leq C^{-2} p^{-k} \mathbf{E}\eta^2 \tag{5.5}$$

where by (3.7), $C^{-2}\mathbf{E}\eta^2 < \infty$. Hence for

$$C_r = \left\{x \in \mathbf{R}^{\mathbf{N}}: \sup_{s \geq k} \left| \sum_{j=k}^{s} x_j \right| \leq p^{-k/3} \quad \text{for all } k \geq r \right\}$$

$$\begin{aligned}
\mathbf{P}(\xi^{(u)} \in C_r) &\geq 1 - \sum_{k \geq r} p^{-k/3} \quad \text{for all } u, \\
&\geq 1 - \varepsilon/2
\end{aligned} \tag{5.6}$$

for r suitably large.

Also if

$$B_M = \left\{x \in \mathbf{R}^{\mathbf{N}}: \sup_{s \geq 0} \left| \sum_{j=0}^{s} x_j \right| \leq M \right\}$$

and M is chosen sufficiently large, then by (5.5),

$$\mathbf{P}(\xi^{(u)} \in B_M) \geq 1 - \varepsilon/2 \quad \text{for all } u. \tag{5.7}$$

Clearly (5.6) and (5.7) imply

$$\mathbf{P}(\xi^{(u)} \in B_M \cap C_r) \geq 1 - \varepsilon \quad \text{for all } u,$$

so $\{\xi^{(u)}\}$ is tight and (5.4) follows. To complete the proof of (3.10) we show that for each $\varepsilon > 0$,

$$\lim_{u\to\infty} \limsup_{n\to\infty} \mathbf{P}(d_1(\xi_n^{(u)}, \xi_n) \geq \varepsilon) = 0. \tag{5.8}$$

Now

$$\begin{aligned}\mathbf{P}\left(\sup_{k>u}\left|\sum_{j=u+1}^{k} \xi_{nj}\right| \geq \varepsilon\right) &\leq \varepsilon^{-2} \sum_{j=u+1}^{n} \mathbf{E}X_{n-j}^2/(a^2 s_n^2)\\ &= \varepsilon^{-2} s_{n-u-1}^2 (a s_n)^{-2}\\ &\to \varepsilon^{-2} a^{-2} p^{-u-1} \quad \text{as } n\to\infty,\end{aligned}$$

and thus

$$\lim_{u\to\infty} \limsup_{n\to\infty} \mathbf{P}(d_1(\xi_n^{(u)}, \xi_n) \geq \varepsilon) = \lim_{n\to\infty} \varepsilon^{-2} a^{-2} p^{-u-1} = 0$$

as required.

To prove (3.11) we again require a round-about route to obtain tightness. From the proof of the tightness of the sequence $\{\xi^{(u)}\}$ we may conclude that there exist r and M such that

$$\mathbf{P}(\xi \in B_M \cap C_r) > 1 - \varepsilon/2$$

and by the existence of such B_M and C_r for each ξ_n and the convergence of $\{\xi_n\}$ to ξ just proved there exist r and M for each $\varepsilon > 0$ such that

$$\mathbf{P}(\xi_n \in B_M \cap C_r) > 1 - \varepsilon/2 \quad \text{for all } n. \tag{5.9}$$

Also, since

$$s_n^{-2} V_n^2 \xrightarrow{L_1} \eta^2,$$

there exists b such that

$$\mathbf{P}(s_n^{-1} V_n < b) > 1 - \varepsilon/2 \quad \text{for all } n.$$

We have

$$\begin{aligned}\{\xi_n \in B_M\} &= \left\{\sup_{s\geq 0}\left|\sum_{j=0}^{s} \xi_{nj}\right| \leq M\right\}\\ &= \left\{\sup_{s\geq 0}\left|\sum_{j=0}^{s} \zeta_{nj}\right| \leq M V_n/s_n\right\}\end{aligned}$$

and similarly

$$\{\xi_n \in C_r\} = \left\{\sup_{s\geq k}\left|\sum_{j=k}^{s} \zeta_{nj}\right| \leq p^{-k/3} V_n/s_n \text{ for all } k \geq r\right\}$$

so that choosing $M_0 = Mb$ and $r_0 = r + l$ where l is such that $p^{-l/3}b \leqslant 1$

$$\mathbf{P}(\zeta_n \in C_{r_0} \cap B_{M_0}) > 1 - \varepsilon \quad \text{for all } n. \tag{5.10}$$

Thus $\{\zeta_n\}$ is tight and Theorem 1 is proved.

6. A branching process example

Let $Z_0 = 1, Z_1, Z_2, \ldots$ denote a super-critical Galton–Watson branching process with $1 < \mathbf{E} Z_1 = m < \infty$ and $0 < \operatorname{Var} Z_1 = \sigma^2 < \infty$. It is easily shown that $S_0 = 0$, $S_n = \sum_{k=1}^{n} (Z_k - mZ_{k-1})$ is a MG when $\mathscr{F}_n = \sigma$-field generated by $Z_0, Z_1, \ldots, Z_n$. The conditional variance

$$V_n^2 = \sum_{k=1}^{n} \mathbf{E}\{[Z_k - mZ_{k-1}]^2 \mid \mathscr{F}_{k-1}\} = \sigma^{-2} \sum_{k=0}^{n-1} Z_k$$

and the variance

$$s_n^2 = \mathbf{E}\, S_n^2 = \sigma^{-2}(m^n - 1)(m-1)^{-1}.$$

It is known (e.g. Athreya and Ney [1] p. 9) that under the conditions above

$$Z_n / m^n \xrightarrow{\text{a.s.}} W$$

where W is non-negative and $\mathbf{E}\, W = 1$. From the Toeplitz Lemma we may conclude that

$$s_n^{-2} V_n^2 \xrightarrow{\text{a.s.}} W$$

and since $\mathbf{E}\, W = 1$, then that

$$s_n^{-2} V_n^2 \xrightarrow{L_1} W. \tag{6.1}$$

Considering the expression for s_n^2 we have also that

$$s_n^{-2} s_{n-r}^2 \to m^{-r} \quad \text{for } r = 0, 1, 2, \ldots. \tag{6.2}$$

Let $\phi(\cdot)$ be the characteristic function of $(Z_1 - m)/\sigma$. Then by the central limit theorem

$$[\phi(tn^{-\frac{1}{2}})]^n \to \exp(-t^2/2)$$

and hence, since

$$Z_n \xrightarrow{\text{a.s.}} \infty \quad \text{on } \{W > 0\},$$

$$[\phi(tZ_{n-1})]^{Z_{n-1}} \xrightarrow{\text{a.s.}} \exp(-t^2/2) \quad \text{on } \{W > 0\}.$$

Now $(Z_n - mZ_{n-1})/\sigma$ may be expressed as a sum of Z_{n-1} variables distributed independently as $(Z_1 - m)/\sigma$, and independently of Z_{n-1}. Thus

$$\mathbf{E}\{e^{it(Z_n - mZ_{n-1})/(Z_{n-1}\sigma)} \mid \mathcal{F}_{n-1}\} = [\phi(tZ_{n-1})]^{Z_{n-1}} \xrightarrow{\text{a.s.}} \exp(-t^2/2) \quad \text{on } \{W>0\}$$

and consequently, since

$$Z_{n-1}^{-1} \sum_{k=0}^{n-1} Z_{k-1} \xrightarrow{\text{a.s.}} m(m-1)^{-1} \quad \text{on } \{W>0\}$$

$$\mathbf{E}\{e^{it(Z_n - mZ_{n-1})/(m^{-\frac{1}{2}}(m-1)^{\frac{1}{2}}V_n)} \mid \mathcal{F}_{n-1}\} \xrightarrow{\text{a.s.}} \exp(-t^2/2) \quad \text{on } \{W>0\}. \tag{6.3}$$

Define

$$\psi_n = (\psi_{n0}, \psi_{n1}, \dots)$$

where

$$\psi_{nj} = \begin{cases} (Z_{n-j} - mZ_{n-j-1})/(m^{-\frac{1}{2}}(m-1)^{\frac{1}{2}}S_n), & j = 0, 1, \dots, n, \\ 0, & j > n \end{cases}$$

and

$$\varphi_n = (\varphi_{n0}, \varphi_{n1}, \dots),$$

where

$$\varphi_{nj} = \begin{cases} (Z_{n-j} - mZ_{n-j-1})/(m^{-\frac{1}{2}}(m-1)^{\frac{1}{2}}V_n), & j = 0, 1, \dots, n, \\ 0, & j > n. \end{cases}$$

Also, let

$$\psi = (W^{\frac{1}{2}}Y_0, W^{\frac{1}{2}}Y_1, \dots)$$

where Y_j are distributed mutually independently and independently of W with Y_j being normal with mean zero and variance m^{-j}. Further, set

$$\psi = (Y_0, Y_1, \dots).$$

As a direct application of Theorem 1, from (6.1) (6.2) and (6.3), we have

Theorem 2. *For any $D \in \mathcal{D}_1$ with $\mathbf{P}(\psi \in \partial D) = 0$ and any $A \in \mathcal{A}$*

$$\mathbf{P}(\{\psi_n \in D\} \cap A \mid W>0) \to \mathbf{P}(\psi \in D)\mathbf{P}(A \mid W>0) \tag{6.4}$$

and for any $D \in \mathcal{D}_1$ with $\mathbf{P}(\psi \in \partial D) = 0$ and any $A \in \mathcal{A}$

$$\mathbf{P}(\{\psi_n \in D\} \cap A \mid W>0) \to \mathbf{P}(\psi \in D)\mathbf{P}(A \mid W>0). \tag{6.5}$$

Corollary 1. *For any x a continuity point of $W^{\frac{1}{2}}Y_1$ and any $A \in \mathcal{A}$*

$$\mathbf{P}(\{\sigma^{-1}m^{(n-1)/2}(Z_n/Z_{n-1} - m) \leq x\} \cap A \mid Z_n > 0)$$
$$\to \mathbf{P}(W^{\frac{1}{2}}Y_1 \leq x)\mathbf{P}(A \mid W>0). \tag{6.6}$$

For any x and any $A \in \mathcal{A}$,

$$\mathbf{P}(\{\sigma^{-1}Z_{n-1}^{\frac{1}{2}}(Z_n/Z_{n-1}-m)\leq x\}\cap A \mid Z_n>0)$$
$$\to \mathbf{P}(Y_1\leq x)\mathbf{P}(A \mid W>0). \tag{6.7}$$

Proof. Apply the Continuous Mapping Theorem, Theorem 5.1 of Billingsley [2] to (6.4) and (6.5) with the mapping

$$h(x)=x_0.$$

Obtain the conditioning with $\{Z_n>0\}$ rather than $\{W>0\}$ by the argument in Dion [4] p. 690.

Corollary 2. *For any x a continuity point of $W^{\frac{1}{2}}Y_1$ and any $A\in\mathcal{A}$,*

$$\mathbf{P}\left(\left\{\sigma^{-1}m^{(n-1)/2}(m-1)^{-\frac{1}{2}}\left(\sum_{j=1}^{n}Z_j\Big/\sum_{j=0}^{n-1}Z_j-m\right)\leq x\right\}\cap A \mid Z_n>0\right)$$
$$\to \mathbf{P}(W^{\frac{1}{2}}Y_1\leq x)\mathbf{P}(A \mid W>0). \tag{6.8}$$

For any x and any $A\in\mathcal{A}$

$$\mathbf{P}\left(\left\{\sigma^{-1}\left(\sum_{j=0}^{n-1}Z_j\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}Z_j\Big/\sum_{j=0}^{n-1}Z_j-m\right)\leq x\right\}\cap A \mid Z_n>0\right)$$
$$\to \mathbf{P}(Y_1\leq x)\mathbf{P}(A \mid W>0). \tag{6.9}$$

Proof. Use the mapping

$$h(x)=\limsup_{n\to\infty}\sum_{j=0}^{n}x_j.$$

Corollaries 1 and 2 are central limit theorems for the Lotka–Nagaev and Harris estimators of the mean m of the offspring distribution of the branching process. These results have previously been obtained by Dion [4].

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